Supplementary material for paper "Robust Modeling of Constant Mean Curvature Surfaces"

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Derivation of the gradient of the extended CVT (ECVT) energy in Eq. 10

Given a facet with three vertices z_1, z_2, z_3 as seed points, the ECVT energy on this facet is given by

$$F(z_1, z_2, z_3) = N \sum_{i=1}^{3} \int_{V_i} \|y - z_i\|^2 d\sigma$$

where N is the number of vertices of the mesh, V_i the Voronoi cell of z_i on this facet. For the convenience of discussion, we may omit the coefficient N in the following derivation because it is a constant.

Now we want to find the gradient $\frac{\partial F}{\partial z_1}$. This facet is in 3D Euclidean space. If we move z_1 along the normal of this facet for an infinitely small distance, the energy will not change at all. So we know that the gradient vector is on the plane of the facet. Now we could put this problem in a 2D setting knowing that the gradient will be the same as in 3D setting. The 2D setting means that the three vertices are on 2D plane. z_1 moves on the 2D plane.

Next we follow a similar approach which is used in [Du et al., 1999] to derive the gradient of standard CVT. We rewrite the energy as $F(z_1) = H(z_1, \Delta)$, where H is a function that maps the point z_1 and the triangle Δ to the energy of CVT. The triangle is a variable that depends on z_1. We differentiate H over z_1.

$$\frac{dH}{dz_1} = \frac{\partial H}{\partial z_1} + \frac{\partial H}{\partial \Delta} \frac{\partial \Delta}{\partial z_1}$$

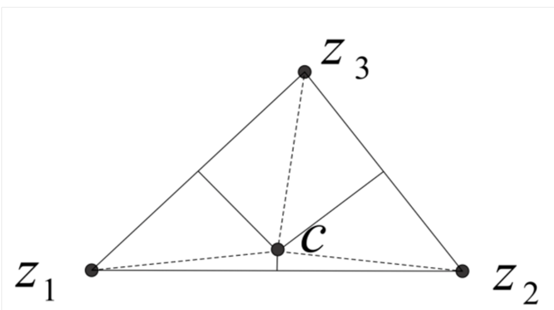

Figure 1: The Voronoi cells on a triangle from which ECVT energy is defined.

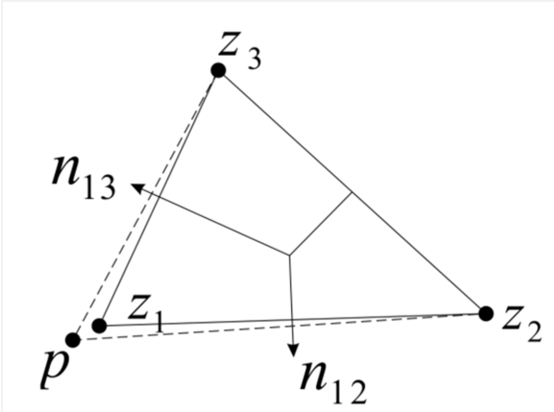

Figure 2: A perturb of the triangle vertex.

The first term $\frac{\partial H}{\partial z_1}$ means the variance of H when the triangle is fixed and only the seed point z_1 changes. It is simply the standard CVT gradient:

$$\frac{\partial H}{\partial z_1} = 2m_1(z_1 - c_1)$$

The second term means the variance of H when the triangle changes and the three seed points are fixed. Furthermore, only the vertex that coincides with z_1 could change. Suppose this vertex is denoted as p. Then this term is equal to $\frac{\partial H}{\partial p}\frac{\partial p}{\partial z_1}$ and $\frac{\partial p}{\partial z_1} = I$. So we need to find out $\frac{\partial H}{\partial p}$.

We find $\frac{\partial H}{\partial p}$ by computing ΔH when there is a Δp. Fig. 2 shows the scenario. We could see that only the two edges pz_2 and pz_3 produce changes of energy H when p is perturbed a bit. To be more specific, the perturbation of pz_2 produces ΔH_{12} for the energy defined over Voronoi region of z_1, and produces ΔH_{21} for the energy over Voronoi region of z_2. Similarly, pz_3 leads to ΔH_{13} for z_1, and ΔH_{31} for z_3. The four energy differences are defined below:

$$\Delta H_{12} = \int_{\frac{z_1+z_2}{2}}^{z_1} \|y - z_1\|^2 \left(1 + \Delta p \cdot \frac{z_1 - z_2}{\|z_1 - z_2\|}\right) dy \frac{\|y - z_2\|}{\|z_1 - z_2\|}(\Delta p \cdot n_{12})$$

where n_{12} is the unit normal vector orthogonal to z_1z_2. The integrand means that for each small line segment on the edge z_2z_1, it expands to a small rectangle when there is a Δp. The value $\|y - z_1\|^2$ is assumed not to vary on such a small patch. And the area of the rectangle is $\left(1 + \Delta p \cdot \frac{z_1 - z_2}{\|z_1 - z_2\|}\right) dy \frac{\|y - z_2\|}{\|z_1 - z_2\|}(\Delta p \cdot n_{12})$. The first part is length of the segment after expansion. The second part is the orthogonal edge length. For ΔH_{12} the interval would be from $\frac{z_1+z_2}{2}$ to z_1. Similarly, the other three energy differences are:

$$\Delta H_{13} = \int_{\frac{z_1+z_3}{2}}^{z_1} \|y - z_1\|^2 \left(1 + \Delta p \cdot \frac{z_1 - z_3}{\|z_1 - z_3\|}\right) dy \frac{\|y - z_3\|}{\|z_1 - z_3\|}(\Delta p \cdot n_{13})$$

$$\Delta H_{21} = \int_{z_2}^{\frac{z_1+z_2}{2}} \|y - z_2\|^2 \left(1 + \Delta p \cdot \frac{z_1 - z_2}{\|z_1 - z_2\|}\right) dy \frac{\|y - z_2\|}{\|z_1 - z_2\|}(\Delta p \cdot n_{12})$$

$$\Delta H_{31} = \int_{z_3}^{\frac{z_1+z_3}{2}} \|y - z_3\|^2 \left(1 + \Delta p \cdot \frac{z_1 - z_3}{\|z_1 - z_3\|}\right) dy \frac{\|y - z_3\|}{\|z_1 - z_3\|}(\Delta p \cdot n_{13})$$

Then we find that

$$\lim_{\|\Delta p\|\to 0} \frac{\Delta H_{12} + \Delta H_{13} + \Delta H_{21} + \Delta H_{31}}{\Delta p} = \frac{1}{24}\left(\|z_1 - z_2\|^3 n_{12} + \|z_1 - z_3\|^3 n_{13}\right)$$

So the gradient of CVT is

$$\frac{dH}{dz_1} = 2m_1(z_1 - c_1) + \frac{1}{24}\left(\|z_1 - z_2\|^3 n_{12} + \|z_1 - z_3\|^3 n_{13}\right).$$

By summing up the gradients on the 1-ring neighborhood facets of a vertex, and putting back the coefficient N, we get the gradient equation of a vertex at Eq. 10.

References

[Du et al., 1999] Du, Q., Faber, V., and Gunzburger, M. (1999). Centroidal Voronoi tessellations: Applications and algorithms. *SIAM Review*, 41(4):637–676.